

# Spectral Properties of Faddeev Equations in Differential Form

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## 1 Introduction

Faddeev equations in differential form were introduced by H.P. Noyes and H. Fiedelday in 1968 [1]

$$(H_0 - E)\varphi_\alpha + V_\alpha \sum_{\beta=1}^3 \varphi_\beta = 0, \quad (1)$$

and since that time are used extensively as for investigating theoretical aspects of the three-body problem as well as for numerical solutions of three-body bound-state and scattering state problems. The simple formula

$$\sum_{\beta=1}^3 \varphi_\beta = \Psi$$

allows one to obtain the solution to the three-body Schrödinger equation

$$(H_0 + \sum_{\beta=1}^3 V_\beta - E)\Psi = 0$$

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in the case when

$$\sum_{\beta=1}^3 \varphi_{\beta} \neq 0. \quad (2)$$

Such solutions of (1) can be called **physical**. The proper asymptotic boundary conditions should be added to Eqs. (1) in order to guarantee (2). This conditions were studied by many authors and are well known [2]. So that, I will not discuss them here.

On the other hand, Eqs. (1) themselves allow solutions of the different type (to physical ones) with the property

$$\sum_{\beta=1}^3 \varphi_{\beta} = 0.$$

This solutions can be constructed explicitly and have the form

$$\varphi_{\alpha} = \sigma_{\alpha} \phi^0,$$

where  $\phi^0$  is an eigenfunction of operator  $H_0$ :

$$H_0 \phi^0 = E^0 \phi^0$$

and  $\sigma_{\alpha}$ ,  $\alpha = 1, 2, 3$  are numbers such that  $\sum_{\alpha=1}^3 \sigma_{\alpha} = 0$ . The solutions of this type can be called **spurious** or **ghost**, because they do not correspond to any three-body system and do not contain any information about interactions between particles. First observation of the existence of spurious solutions was made in ref. [3]. Some spurious solutions corresponding to particular values of the total angular momentum were found in refs. [4], [5]. All the spurious solutions on subspaces with fixed total angular momentum were constructed in ref. [6].

So that, there exist at least two types of solutions to Eqs. (1) corresponding to real energy:

**physical** ones with the property  $\sum_{\beta=1}^3 \varphi_{\beta} \neq 0$ ,

**spurious** ones with the property  $\sum_{\beta=1}^3 \varphi_{\beta} = 0$ .

The QUESTION is do these solutions form the complete set or there could be exist solutions of different type which do not belong to physical and spurious

classes. The ANSWER is not so evident because the operator corresponding to Eqs. (1) is not selfadjoint and moreover even symmetrical:

$$\mathbf{H} = \begin{pmatrix} H_0 & 0 & 0 \\ 0 & H_0 & 0 \\ 0 & 0 & H_0 \end{pmatrix} + \begin{pmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \mathbf{H}_0 + \mathbf{V}\mathbf{X}, \quad (3)$$

and, in principle, this operator can have not real eigenvalues even the ingredients  $H_0$ ,  $V_\alpha$  and three-body Hamiltonian  $H = H_0 + \sum_{\beta=1}^3 V_\beta$  are selfadjoint operators.

In this report I will answer on the QUESTION and will give a classification of eigenfunctions of the operator  $\mathbf{H}$  and its adjoint. This report is based on refs. [7], [8].

## 2 Faddeev operator and its ajoint

Let us consider the Hilbert space  $\mathcal{H}$  of three component vectors  $F = \{f_1, f_2, f_3\}$ . The operator  $\mathbf{H}$  acts in  $\mathcal{H}$  according to the formula

$$(\mathbf{H}F)_\alpha = H_0 f_\alpha + V_\alpha \sum_{\beta} f_\beta. \quad (4)$$

The adjoint  $\mathbf{H}^*$  is defined as

$$\mathbf{H}^* = \mathbf{H}_0 + \mathbf{X}\mathbf{V} = \begin{pmatrix} H_0 & 0 & 0 \\ 0 & H_0 & 0 \\ 0 & 0 & H_0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{pmatrix}$$

and acts as follows

$$(\mathbf{H}^*G)_\alpha = H_0 g_\alpha + \sum_{\beta} V_\beta g_\beta. \quad (5)$$

The equations for eigenvectors of operators  $\mathbf{H}$  and  $\mathbf{H}^*$

$$\mathbf{H}\Phi = E\Phi, \quad \mathbf{H}^*\Psi = E\Psi$$

in components have the form

$$H_0 \varphi_\alpha + V_\alpha \sum_{\beta=1}^3 \varphi_\beta = E \varphi_\alpha,$$

$$H_0\psi_\alpha + \sum_{\beta=1}^3 V_\beta\psi_\beta = E\psi_\alpha.$$

The first one coincides to the Faddeev equations (1) and the second one has the direct connection to the so called triad of Lippmann-Schwinger equations [9].

It follows directly from the definitions (4) and (5) that operators  $\mathbf{H}$  and  $\mathbf{H}^*$  have the following invariant subspaces:  
for  $\mathbf{H}$

$$\mathcal{H}_s = \{F \in \mathcal{H}_s : \sum_{\alpha} f_{\alpha} = 0\},$$

for  $\mathbf{H}^*$

$$\mathcal{H}_p^* = \{G \in \mathcal{H}_p^* : g_1 = g_2 = g_3 = g\}.$$

It is worth to notice that operators  $\mathbf{H}$  and  $\mathbf{H}^*$  on the subspaces  $\mathcal{H}_s$  and  $\mathcal{H}_p^*$  act as free Hamiltonian  $H_0$  and three-body Hamiltonian  $H$ , respectively:

$$(\mathbf{H}F)_{\alpha} = H_0 f_{\alpha} \quad , \text{ if } F \in \mathcal{H}_s,$$

$$(\mathbf{H}^*G)_{\alpha} = Hg = H_0g + \sum_{\beta} V_{\beta}g \quad , \text{ if } G \in \mathcal{H}_p^*.$$

As a consequence the spectrum of  $\mathbf{H}$  on  $\mathcal{H}_s$  coincides to the spectrum of  $H_0$  and the spectrum of  $\mathbf{H}^*$  on  $\mathcal{H}^*$  does to the spectrum of three-body Hamiltonian  $H$ .

In order to describe eigenfunctions of operators  $\mathbf{H}$  and  $\mathbf{H}^*$  let us introduce the resolvents

$$\mathbf{R}(z) = (\mathbf{H} - z)^{-1},$$

$$\mathbf{R}^*(z) = (\mathbf{H}^* - z)^{-1}.$$

The components of these resolvents can be expressed through the resolvent of three-body Hamiltonian and free Hamiltonian as follows

$$R_{\alpha\beta}(z) = R_0(z)\delta_{\alpha\beta} - R_0(z)V_{\alpha}R(z), \quad (6)$$

$$R_{\alpha\beta}^*(z) = R_0(z)\delta_{\alpha\beta} - R(z)V_{\beta}R_0(z). \quad (7)$$

Here

$$R(z) = (H - z)^{-1} = (H_0 + \sum_{\beta} V_{\beta} - z)^{-1}, \quad R_0(z) = (H_0 - z)^{-1}.$$

It is worth to note that the components of resolvents obey the following Faddeev equations

$$R_{\alpha\beta}(z) = R_\alpha(z)\delta_{\alpha\beta} - R_\alpha(z)V_\alpha \sum_{\gamma \neq \alpha} R_{\gamma\beta}(z), \quad (8)$$

$$R_{\alpha\beta}^*(z) = R_\alpha(z)\delta_{\alpha\beta} - R_\alpha(z) \sum_{\gamma \neq \alpha} V_\gamma R_{\gamma\beta}^*(z). \quad (9)$$

Here  $R_\alpha(z) = (H_0 + V_\alpha - z)^{-1}$  is the two-body resolvent for the pair  $\alpha$  in the three-body space.

In order to proceed it is convenient to introduce the spectral representation for the resolvent of three-body Hamiltonian

$$R(z) = \sum_{E_i} \frac{|\psi^i\rangle\langle\psi^i|}{E_i - z} + \sum_{\gamma} \int dp_{\gamma} \frac{|\psi^{\gamma}(p_{\gamma})\rangle\langle\psi^{\gamma}(p_{\gamma})|}{p_{\gamma}^2 - z} + \int dP \frac{|\psi^0(P)\rangle\langle\psi^0(P)|}{P^2 - z}.$$

It is implied here that the system of eigenfunctions of the operator  $H$  is complete *i.e.*,

$$I = \sum_i |\psi^i\rangle\langle\psi^i| + \sum_{\gamma} \int dp_{\gamma} |\psi^{\gamma}(p_{\gamma})\rangle\langle\psi^{\gamma}(p_{\gamma})| + \int dP |\psi^0(P)\rangle\langle\psi^0(P)|.$$

Introducing this representation into (6) and (7) one arrives to the spectral representations for components  $R_{\alpha\beta}(z)$ :

$$\begin{aligned} R_{\alpha\beta}(z) = & \sum_{E_i} \frac{|\psi_{\alpha}^i\rangle\langle\psi^i|}{E_i - z} + \sum_{\gamma} \int dp_{\gamma} \frac{|\psi_{\alpha}^{\gamma}(p_{\gamma})\rangle\langle\psi^{\gamma}(p_{\gamma})|}{p_{\gamma}^2 - z} + \\ & \int dP \frac{|\psi_{\alpha}^{10}(P)\rangle\langle\psi^0(P)|}{P^2 - z} + \sum_{k=1}^2 \int dP \frac{|u_{\alpha}^k(P)\rangle\langle w_{\beta}^k(P)|}{P^2 - z}. \end{aligned} \quad (10)$$

Here  $\psi_{\alpha}^i$ ,  $\psi_{\alpha}^{\gamma}(p_{\gamma})$  and  $\psi_{\alpha}^{10}(P)$  are the Faddeev components of eigenfunctions of three-body Hamiltonian:

$$\begin{aligned} \psi_{\alpha}^i &= -R_0(E_i)V_{\alpha}\psi^i, \\ \psi_{\alpha}^{\gamma}(p_{\gamma}) &= -R_0(\varepsilon_{\gamma} + p_{\gamma}^2 + i0)V_{\alpha}\psi^{\gamma}(p_{\gamma}), \\ \psi_{\alpha}^{10}(P) &= \delta_{\alpha 1}\phi^0(P) - R_0(P^2 + i0)V_{\alpha}\psi^0(P), \end{aligned}$$

where  $\phi^0(P)$  is an eigenfunction of the free Hamiltonian:

$$H_0\phi^0(P) = P^2\phi^0(P).$$

A new feature in (10) is the appearance of the last term related to the spurious solutions of Faddeev equations and its adjoint. The explicit formulas for the spurious eigenfunctions  $u_\alpha^k(P)$  of  $\mathbf{H}$  are of the form

$$u_\alpha^k(P) = \sigma_\alpha^k\phi^0(P),$$

where  $\sigma_\alpha^k$ ,  $k = 1, 2$ , are the components of two noncollinear vectors from  $\mathbf{R}^3$  lying on the plane  $\sum_\alpha \sigma_\alpha = 0$ . The spurious eigenfunctions  $w_\beta^k(P)$  of  $\mathbf{H}^*$  can be expressed by the formula

$$w_\beta^k(P) = \theta_\beta^k\phi^0(P) - \sum_\alpha [\mathcal{P}_p^*]_{\beta\alpha} \theta_\alpha^k\phi^0(P),$$

where

$$[\mathcal{P}_p^*]_{\beta\alpha} = \sum_i |\psi^i\rangle\langle\psi_\alpha^i| + \sum_\gamma \int dp'_\gamma |\psi^\gamma(p'_\gamma)\rangle\langle\psi_\alpha^\gamma(p'_\gamma)| + \int dP' |\psi^0(P')\rangle\langle\psi_\alpha^{01}(P')|.$$

Here the vectors  $\theta^k \in \mathbf{R}^3$  are defined by following biorthogonality conditions

$$\sum_\alpha \theta_\alpha^i \sigma_\alpha^j = \delta_{ij}, \quad i, j = 0, 1, 2,$$

with  $\sigma_\alpha^0 = \delta_{\alpha 1}$  and  $\theta_\alpha^0 = 1$ .

For the components of resolvent  $R_{\alpha\beta}^*(z)$  one can obtain the similar to (10) formula

$$\begin{aligned} R_{\alpha\beta}^*(z) = & \sum_{E_i} \frac{|\psi^i\rangle\langle\psi_\beta^i|}{E_i - z} + \sum_\gamma \int dp_\gamma \frac{|\psi^\gamma(p_\gamma)\rangle\langle\psi_\beta^\gamma(p_\gamma)|}{p_\gamma^2 - z} + \int dP \frac{|\psi^0(P)\rangle\langle\psi_\beta^{10}(P)|}{P^2 - z} + \\ & + \sum_{k=1}^2 \int dP \frac{|w_\alpha^k(P)\rangle\langle u_\beta^k(P)|}{P^2 - z}. \end{aligned} \quad (11)$$

It follows from (10) and (11) that operators  $\mathbf{H}$  and  $\mathbf{H}^*$  have the following system of eigenfunctions:

$$\{ \Phi^i, \Phi^\gamma(p_\gamma), \Phi^{10}(P) \text{ and } U^k(P) \}$$

$$\mathbf{H}\Phi^i = E_i\Phi^i,$$

$$\mathbf{H}\Phi^\gamma(p_\gamma) = (\varepsilon_\gamma + p_\gamma^2)\Phi^\gamma(p_\gamma),$$

$$\mathbf{H}\Phi^{10}(P) = P^2\Phi^{10}(P),$$

$$\mathbf{H}U^k(P) = P^2U^k, \quad k = 1, 2;$$

$$\{ \Psi^i, \Psi^\gamma(p_\gamma), \Psi^{10}(P) \text{ and } W^k(P) \}$$

$$\mathbf{H}^*\Psi^i = E_i\Psi^i,$$

$$\mathbf{H}^*\Psi^\gamma(p_\gamma) = (\varepsilon_\gamma + p_\gamma^2)\Psi^\gamma(p_\gamma),$$

$$\mathbf{H}^*\Psi^{10}(P) = P^2\Psi^{10}(P),$$

$$\mathbf{H}^*W^k(P) = P^2W^k, \quad k = 1, 2,$$

with components of physical eigenfunctions:

$$\phi_\alpha^i = -R_0(E_i)V_\alpha\psi^i,$$

$$\phi_\alpha^\gamma(p_\gamma) = -R_0(\varepsilon_\gamma + p_\gamma^2 + i0)V_\alpha\psi^\gamma(p_\gamma),$$

$$\phi_\alpha^{10}(P) = \delta_{\alpha 1}\phi^0(P) - R_0(P^2 + i0)V_\alpha\psi^0(P),$$

for  $\mathbf{H}$  and with components for physical eigenfunctions:

$$\psi_\alpha^i = \psi^i,$$

$$\psi_\alpha^\gamma(p_\gamma) = \psi^\gamma(p_\gamma),$$

$$\psi_\alpha^{10}(P) = \psi^0(P)$$

for  $\mathbf{H}^*$ .

Physical eigenfunctions span the physical subspace of  $\mathcal{H}$ . This subspace can be defined as

$$\mathcal{H}_p = \mathcal{P}_p\mathcal{H},$$

where the projection  $\mathcal{P}_p$  is defined by formula

$$\mathcal{P}_p = \sum_i |\Phi^i\rangle\langle\Psi^i| + \sum_\gamma \int dp_\gamma |\Phi^\gamma(p_\gamma)\rangle\langle\Psi^\gamma(p_\gamma)| + \int dP |\Phi^{10}(P)\rangle\langle\Psi^{10}(P)|.$$

Spurious solutions span the spurious subspace of  $\mathcal{H}$ :

$$\mathcal{H}_s = \mathcal{P}_s \mathcal{H}.$$

where

$$\mathcal{P}_s = \sum_{k=1}^2 \int dP |U^k(P)\rangle \langle W^k(P)|.$$

It follows from construction and completeness of eigenfunctions of three-body Hamiltonian, that physical and spurious subspaces are complete in  $\mathcal{H}$ :

$$\mathcal{H} = \mathcal{H}_p + \mathcal{H}_s.$$

The same is valid for physical and spurious subspaces of operator  $\mathbf{H}^*$ :

$$\mathcal{H} = \mathcal{H}_p^* + \mathcal{H}_s^*,$$

where the subspaces  $\mathcal{H}_p^*$  and  $\mathcal{H}_s^*$  are defined as

$$\mathcal{H}_p^* = \mathcal{P}_p^* \mathcal{H}, \quad \mathcal{H}_s^* = \mathcal{P}_s^* \mathcal{H}.$$

Here the operators  $\mathcal{P}_p^*$  and  $\mathcal{P}_s^*$  are Hilbert space adjoints for  $\mathcal{P}_p$  and  $\mathcal{P}_s$ .

The results described above can be summarized as the following

**Theorem:** *Faddeev operator  $\mathbf{H}$*

$$\mathbf{H} = \begin{pmatrix} H_0 & 0 & 0 \\ 0 & H_0 & 0 \\ 0 & 0 & H_0 \end{pmatrix} + \begin{pmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and its adjoint  $\mathbf{H}^*$

$$\mathbf{H}^* = \begin{pmatrix} H_0 & 0 & 0 \\ 0 & H_0 & 0 \\ 0 & 0 & H_0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{pmatrix}$$

have coinciding spectrums of real eigenvalues

$$\sigma(\mathbf{H}) = \sigma(\mathbf{H}^*) = \sigma(H) \cup \sigma(H_0),$$

where the physical part of the spectrum  $\sigma(H)$  is the spectrum of the three-body Hamiltonian  $H = H_0 + \sum_{\alpha} V_{\alpha}$  and the spurious part  $\sigma(H_0)$  is the spectrum of the free Hamiltonian  $H_0$ . The sets of physical and spurious eigenfunctions are complete and biorthogonal in the sense:

$$\begin{aligned} \mathcal{P}_p + \mathcal{P}_s &= \mathcal{P}_p^* + \mathcal{P}_s^* = I, \\ \mathcal{P}_{p(s)}^2 &= \mathcal{P}_{p(s)}, \quad \mathcal{P}_{p(s)}^{*2} = \mathcal{P}_{p(s)}^*, \quad \mathcal{P}_p \mathcal{P}_s^* = 0, \quad \mathcal{P}_s \mathcal{P}_p^* = 0. \end{aligned}$$

### 3 Extension on CCA equations

It is shown that the matrix operator generated by Faddeev equations in differential form has the (additional to physical) spurious spectrum. The existence of this spectrum strongly relates to the invariant spurious subspace formed by components which sum is equal to zero. The theorem formulated in preceding section can be extended on any matrix operator corresponding to few-body equations for components of wave-function obtained in framework of so called coupled channel array (CCA) method [10] as follows. CCA equations can be written in the matrix form as

$$\mathbf{H}\Phi = E\Phi, \quad (12)$$

where  $\mathbf{H}$  is a  $n \times n$  matrix operator acting in the Hilbert space  $\mathcal{H}$  of vector-functions  $\Phi$  with components  $\phi_1, \phi_2, \dots, \phi_n$  each belonging to few-body system Hilbert space  $h$ . The equivalence of Eq. (12) to the Schrödinger equation  $H\psi = (H_0 + \sum_{\beta} V_{\beta})\psi = E\psi$  by requiring  $\sum_{\alpha} \phi_{\alpha} = \psi$  can be reformulated as the following intertwining property for operators  $\mathbf{H}$  and  $H$

$$\mathcal{S}\mathbf{H} = H\mathcal{S}. \quad (13)$$

Here  $\mathcal{S}$  is the summation operator

$$\mathcal{S}\Phi = \sum_{\alpha} \phi_{\alpha}$$

acting from  $\mathcal{H}$  to  $h$ . Due to (13) the subspace  $\mathcal{H}_s$  formed by spurious vectors such that  $\mathcal{S}\Phi = 0$  is invariant with respect to  $\mathbf{H}$  and as a consequence the operator  $\mathbf{H}$  has the spurious spectrum  $\sigma_s$ . Clearly, that the concrete form of  $\sigma_s$  and of corresponding eigenfunctions depends on the particular form of the matrix operator  $\mathbf{H}$  and is the subject of special investigation.

The physical part  $\sigma_p$  of the spectrum of  $\mathbf{H}$  can be found with adjoint variant of (13)

$$\mathbf{H}^*\mathcal{S}^* = \mathcal{S}^*H, \quad (14)$$

where adjoint  $\mathcal{S}^*$  acts from  $h$  to  $\mathcal{H}$  according to the formula

$$[\mathcal{S}^*\phi]_{\alpha} = \phi.$$

It follows from Eq. (14) that the range  $\mathcal{H}_p^*$  of operator  $\mathcal{S}^*$  consisting of vector-functions with the same components is invariant with respect to  $\mathbf{H}^*$

and the restriction of  $\mathbf{H}^*$  on  $\mathcal{H}_p^*$  is reduced to few-body Hamiltonian  $H$ . So that,  $\sigma_p = \sigma(H)$  and, similarly to the case of the Faddeev operator, the same formula for the spectrums of operators  $\mathbf{H}$  and  $\mathbf{H}^*$  is valid

$$\sigma(\mathbf{H}) = \sigma(\mathbf{H}^*) = \sigma(H) \cup \sigma_s,$$

where  $\sigma(H)$  is the spectrum of few-body Hamiltonian  $H = H_0 + \sum_{\alpha} V_{\alpha}$ .

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